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Asymptotic expansions and acceleration methods for certain logarithmically convergent sequences
(ある種の対数収束数列の漸近展開と加速法)

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1. Introduction.

A sequence (S_n) converging to a limit S is said to be logarithmically convergent if

$$\lim_{n\to\infty} \frac{S_{n+1}-S}{S_n-S} = 1.$$

We denote by $\mathscr{L}$ the set of all logarithmically convergent sequences (logarithmic sequences, for short). It has been proved that there is no algorithm which can accelerate all logarithmic sequences [5]. However almost all logarithmic sequences which occur in applied mathematics can be accelerated by suitable methods. C. Kowalewski [8] has studied what are the accelerable subsets of $\mathscr{L}$. Smith and Ford [13] have reviewed and compared acceleration methods for various series, including logarithmically convergent series. They concluded that Levin's u transform is the best available across-the-board method. Subsequently some effective acceleration methods for logarithmic sequences have been proposed. These methods are faster than the u transform on logarithmic sequences.

In this report we shall test and compare acceleration methods, including new methods, on a wide range of logarithmic sequences.

2. Subsets of $\mathscr{L}$.

Whether an acceleration method works effectively on a given sequence or not depends on the asymptotic expansion of the sequence. Conversely, when we know the type of asymptotic expansion of a sequence we can choose a suitable acceleration method.

In order to test and compare acceleration methods we introduce six subsets of $\mathscr{L}$ as follows:

$$\mathscr{L}_0=\{ (S_n)\in \mathscr{L} \mid S_n = \frac{a_0+a_1n+a_2n^2+\cdots+a_mn^m}{b_0+b_1n+b_2n^2+\cdots+b_mn^m} \quad \forall n \},$$

$$\mathscr{L}_1=\{ (S_n)\in \mathscr{L} \mid S_n \sim S+n^{-1}(c_0+c_1/n+c_2/n^2+\cdots) \},$$

$$\mathscr{L}_2=\{ (S_n)\in \mathscr{L} \mid S_n \sim S+n^{-k}(c_0+c_1/n+c_2/n^2+\cdots) \},$$

$$\mathscr{L}_3=\{ (S_n)\in \mathscr{L} \mid S_n \sim S+n^{\theta}(c_0+c_1/n+c_2/n^2+\cdots) \},$$

$$\mathscr{L}_4=\{ (S_n)\in \mathscr{L} \mid S_n \sim S+c_0n^{\alpha_0}+c_1n^{\alpha_1}+c_2n^{\alpha_2}+\cdots \},$$

$$\mathscr{L}_5=\{ (S_n)\in \mathscr{L} \mid S_n \sim S+\sum_{i,j} c_{i,j}\frac{(\log n)^j}{n^i} \},$$

where m,k are natural numbers, $a_j, b_j, c_j, c_{i,j}, \theta, \alpha_j (i=1,2,\cdots; j=0,1,\cdots)$are constants with $c_0\neq 0$, $\theta<0$ and $0>\alpha_0>\alpha_1>\alpha_2>\cdots$.

Remark. $\mathscr{L}_1\subset\mathscr{L}_2\subset\mathscr{L}_3\subset\mathscr{L}_4$ and $\mathscr{L}_0\subset\mathscr{L}_2\subset\mathscr{L}_5$.

3. Acceleration methods.

Until now various acceleration methods for logarithmic sequences have been proposed. In this report we take up following methods, and divide them into two groups:

I The exponents in the asymptotic expansion are not required.
Lubkin's W transform [10]; the ρ-algorithm (Wynn [15]); the θ-algorithm (Brezinski [3]); Levin's u,v transforms [9]; the generalized ε-algorithm (Vanden Broeck et al.[14]); the automatic generalized ρ-algorithm (Osada [11]); the automatic modified Aitken δ^2-formula.

II The exponents in the asymptotic expansion are required.
The Richardson extrapolation applying to the subsequence (S_m) m=1,2,4,8,$\cdots$; the E-algorithm (Schneider [12];Håvie [7];Brezinski [4]); the modified Aitken δ^2-formula (Drummond [6];Bjørstad et al. [2]); the generalized ρ-algorithm (Drummond [6];Osada [11]).

In the above methods, ρ, θ-algorithms and u,v transforms were taken up by Smith and Ford [13]. Thus we describe the others.

(1) Lubkin's W transform.

The defining equations for Lubkin's W transform are as follows:

Initializations $T_{0,j}=0 \quad j=0,1,\cdots.$

For $i=1,2,\cdots,$

$$T_{i,0}=S_i,$$

$$a_{i,0}=\begin{cases} S_1 & (i=1) \\ S_i-S_{i-1} & (i\geqq 2), \end{cases}$$

$$T_{i,j}=\frac{\dfrac{T_{i+2,j-1}}{a_{i+2,j-1}}-\dfrac{2T_{i+1,j-1}}{a_{i+1,j-1}}+\dfrac{T_{i,j-1}}{a_{i,j-1}}}{\dfrac{1}{a_{i+2,j-1}}-\dfrac{2}{a_{i+1,j-1}}+\dfrac{1}{a_{i,j-1}}} \qquad j=1,2,\cdots,$$

$$a_{i,j}=T_{i,j}-T_{i-1,j} \qquad j=1,2,\cdots.$$

Lubkin [10] has proved that the W transform accelerates each sequence of $\mathscr{L}_3$.

(2) The generalized ε-algorithm.

Vanden Broeck and Schwartz have introduced a one-parameter family of non-linear transformations defined by the following:

For $n=1,2,\cdots,$

$$\varepsilon_n^{(-1)}=0,$$

$$f_n^{(0)}=S_n.$$

For $n=1,2,\cdots$ and $m=0,1,\cdots,$

$$\varepsilon_n^{(m)}=\alpha\,\varepsilon_n^{(m-1)}+\frac{1}{f_{n+1}^{(m)}-f_n^{(m)}} \qquad n>m,$$

$$f_n^{(m+1)}=f_n^{(m)}+\frac{1}{\varepsilon_n^{(m)}-\varepsilon_{n-1}^{(m)}} \qquad n>m+1.$$

When the parameter $\alpha=1$, this algorithm agrees with the ε-algorithm of Wynn. They considered the case $\alpha=-1$. Barber and Hamer [1] have proved that when the generalized ε-algorithm is applied to a sequence satisfying

$$S_n-S=An^{\theta}+o(n^{\theta}) \qquad \text{as } n\to\infty,$$

where $\theta < 0$, then

$$f_n^{(2)} - S = o(n^{\theta}) \qquad \text{as } n \to \infty .$$

(3) The modified Aitken δ^2-formula.

Suppose that a sequence (S_n) belongs to $\mathcal{L}_3$. The defining equations for the modified Aitken δ^2-formula are as follows:

$$s_0^0 = 0,$$

$$s_n^0 = S_n, \qquad \text{for } n = 1, 2, \cdots$$

$$s_n^{k+1} = s_n^k - \frac{2k+1-\theta}{2k-\theta} \frac{(s_{n+1}^k - s_n^k)(s_n^k - s_{n-1}^k)}{s_{n+1}^k - 2s_n^k + s_{n-1}^k}, \quad \text{for } k = 0, 1, \cdots \; n \geqq k+1.$$

Bjørstad, Dahlquist and Grosse [2] have proved that when the modified Aitken δ^2-formula is applied to (S_n) which belongs to $\mathcal{L}_3$,

$$s_n^k - S = O(n^{\theta - 2k}) \qquad \text{as } n \to \infty .$$

(4) The generalized ρ-algorithm.

For a sequence (S_n) which belongs to $\mathcal{L}_3$, the generalized ρ-algorithm is defined as follows:

Put $S_0 = 0$.

For $n = 0, 1, \cdots$,

$$\bar{\rho}_{-1}^{(n)} = 0, \qquad \bar{\rho}_0^{(n)} = S_n,$$

$$\bar{\rho}_j^{(n)} = \bar{\rho}_{j-2}^{(n+1)} + \frac{j-1-\theta}{\bar{\rho}_{j-1}^{(n+1)} - \bar{\rho}_{j-1}^{(n)}}, \qquad j = 1, 2, \cdots .$$

It is obvious that, when $\theta = -1$, the generalized ρ-algorithm agrees with Wynn's ρ-algorithm. It has been proved [11] that

$$\bar{\rho}_{2k}^{(n)} - S = O((n+K)^{\theta - 2k}) \qquad \text{as } n \to \infty .$$

(5) The Richardson extrapolation.

Suppose that a sequence (S_n) satisfies

$$S_n \sim S + c_0 n^{\alpha_0} + c_1 n^{\alpha_1} + c_2 n^{\alpha_2} + \cdots .$$

Then it is known that the subsequence (S_m) $(m = 1, 2, 4, 8, 16, \cdots)$ converges linearly to S, hence the Richardson extrapolation can be applied to the subsequence. Two dimensional array $(T_{i,j})$ is defined as follows:

For $i=0,1,\cdots$,

$$T_{i,0}=S_k,\quad \text{where } k=2^i,$$

$$T_{i,j+1}=T_{i,j}+\frac{T_{i,j}-T_{i-1,j}}{2^{-\alpha j}-1}\qquad j=0,\cdots,i-1.$$

(6) The E-algorithm.

Suppose that a sequence (S_n) satisfies

$$S_n \sim S+c_0g_0(n)+c_1g_1(n)+c_2g_2(n)+\cdots,$$

where $(g_j(n))$ are known sequences. The defining equations of the E-algorithm are as follows:

For $n=1,2,\cdots$,

$$E_0^{(n)}=S_n,$$

$$g_{0,i}^{(n)}=g_i(n)\qquad i=0,1,\cdots.$$

For $k=1,2,\cdots$ and $n=1,2,\cdots$,

$$E_k^{(n)}=\frac{E_{k-1}^{(n)}g_{k-1,k-1}^{(n+1)}-E_{k-1}^{(n+1)}g_{k-1,k-1}^{(n)}}{g_{k-1,k-1}^{(n+1)}-g_{k-1,k-1}^{(n)}}$$

$$g_{k,i}^{(n)}=\frac{g_{k-1,i}^{(n)}g_{k-1,k-1}^{(n+1)}-g_{k-1,i}^{(n+1)}g_{k-1,k-1}^{(n)}}{g_{k-1,k-1}^{(n+1)}-g_{k-1,k-1}^{(n)}}\qquad i=k,k+1,\cdots.$$

(7) Automatic methods.

Both the modified Aitken δ^2-formula and the generalized ρ-algorithm require knowledge of θ. However the value of θ can be estimated. Let

$$T_n=\frac{1}{\Delta\left(\dfrac{S_{n+1}-S_n}{S_{n+1}-2S_n+S_{n-1}}\right)}+1,$$

where Δ is the forward difference operator and $S_0=0$. Bjørstad et al.[2] showed that T_n satisfies the asymptotic expansion

$$T_n \sim \theta+n^{-2}\left(d_0+\frac{d_1}{n}+\frac{d_2}{n^2}+\cdots\right)\qquad \text{as } n\to\infty,$$

where $d_0(\neq 0),d_1,d_2,\cdots$ are constants. This asymptotic expansion shows that the modified Aitken δ^2-formula and the generalized ρ-algorithm can be applied to (T_n).

(7-1) The automatic modified Aitken δ^2-formula.

Suppose that the first n terms $S_1, S_2, \cdots, S_n$ of a sequence which belongs to $\mathscr{L}_3$ are given. Then we define (t_m^k) as follows:

For $m=1$ to $n-2$,

$t_m^0 = T_m$,

For $k=1$ to $\lfloor m/2 \rfloor$,

$$t_{m-k}^k = t_{m-k}^{k-1} - \frac{2k+1}{2k} \frac{(t_{m-k+1}^{k-1} - t_{m-k}^{k-1})(t_{m-k}^{k-1} - t_{m-k-1}^{k-1})}{t_{m-k+1}^{k-1} - 2t_{m-k}^{k-1} + t_{m-k-1}^{k-1}}.$$

Then we put

$\theta_{n-2} = t_{n-k-2}^k$, where $k = \lfloor (n-2)/2 \rfloor$.

Substituting θ_{n-2} for θ in the definition of the modified Aitken δ^2-formula, we can obtain the automatic modified Aitken δ^2-formula:

For $m=1$ to n,

$s_{n,m}^0 = S_m$,

For $k=1$ to $\lfloor m/2 \rfloor$,

$$s_{n,m-k}^k = s_{n,m-k}^{k-1} - \frac{2k-1-\theta_{n-2}}{2k-2-\theta_{n-2}}$$

$$\times \frac{(s_{n,m-k+1}^{k-1} - s_{n,m-k}^{k-1})(s_{n,m-k}^{k-1} - s_{n,m-1-1}^{k-1})}{s_{n,m-k+1}^{k-1} - 2s_{n,m-k}^{k-1} + s_{n,m-k-1}^{k-1}}.$$

For a given tolerance ε, this scheme is stopped if n is even and

$|s_{n,n-k}^k - s_{n,n-k}^{k-1}| < \varepsilon$,

or if n is odd and

$|s_{n,n-k}^k - s_{n,n-k-1}^k| < \varepsilon$,

where $k = \lfloor n/2 \rfloor$.

(7-2) The automatic generalized ρ-algorithm.

Suppose that the first n terms $S_1, S_2, \cdots, S_n$ of a sequence which belongs to $\mathscr{L}_3$ are given. First we estimate the exponent θ by applying the generalized ρ-algorithm to (T_n).

Initialization $\bar{\rho}_0^{(0)} = 0,\ S_0 = 0.$

For $m = 1$ to $n-2$,

$$\bar{\rho}_{-1}^{(m)} = 0,$$

$$\bar{\rho}_0^{(m)} = \cfrac{1}{\cfrac{S_{m+2} - S_{m+1}}{S_{m+2} - 2S_{m+1} + S_m} - \cfrac{S_{m+1} - S_m}{S_{m+1} - 2S_m + S_{m-1}}} + 1,$$

For $j = 1$ to m,

$$\bar{\rho}_j^{(m-j)} = \bar{\rho}_{j-2}^{(m-j+1)} + \frac{j+1}{\bar{\rho}_{j-1}^{(m-j+1)} - \bar{\rho}_{j-1}^{(m-j)}}.$$

Then the automatic generalized ρ-algorithm is as follows:

Let

$$\theta_{n-2} = \begin{cases} \bar{\rho}_{n-3}^{(1)}, & \text{if } n \text{ is odd}, \\ \bar{\rho}_{n-2}^{(0)}, & \text{if } n \text{ is even}. \end{cases}$$

For $r = 0$ to n,

$$\bar{\rho}_{n,-1}^{(r)} = 0,$$

$$\bar{\rho}_{n,0}^{(r)} = S_r,$$

For $j = 1$ to r,

$$\bar{\rho}_{n,j}^{(r-j)} = \bar{\rho}_{n,j-2}^{(r-j+1)} + \frac{j-1-\theta_{n-2}}{\bar{\rho}_{n,j-1}^{(r-j+1)} - \bar{\rho}_{n,j-1}^{(r-j)}}.$$

For a given tolerance ε, this scheme is stopped if n is even and

$$|\bar{\rho}_{n,n}^{(0)} - \bar{\rho}_{n,n-2}^{(1)}| < \varepsilon,$$

or if n is odd and

$$|\bar{\rho}_{n,n-1}^{(1)} - \bar{\rho}_{n,n-1}^{(0)}| < \varepsilon.$$

4. Test problems.

All Smith and Ford's test series [13] for logarithmically convergent belong to $\mathcal{L}_3$. Similarly, almost logarithmic sequences and series which have been taken up by other authors belong to

$\mathcal{L}_3$. In this report we use a wide range of logarithmic sequences or series.

Our logarithmically convergent test sequences are shown in Table 1 and test series are shown in Table 2.

Table 1. Test sequences

subset	sequence	limit	θ
$\mathcal{L}_0$	$-\dfrac{2n^2+4n+2}{2n^2+4n+1}$	-1	-2
$\mathcal{L}_1$	$(1+\dfrac{1}{n})^n$	e	-1
$\mathcal{L}_2$	$(1+\dfrac{1}{n^3})^n$	1	-2
$\mathcal{L}_3$	$\dfrac{1}{\sqrt{n}+\sqrt{(n+1)}}$	0	-0.5
$\mathcal{L}_4$	$(1+\dfrac{1}{\sqrt[3]{n}})^{1/2}$	1	(*1)
$\mathcal{L}_5$	$(1+n)^{1/n}$	1	(*2)

*1 $\quad (1+\dfrac{1}{\sqrt[3]{n}})^{1/2} \sim 1+\dfrac{1}{2}n^{-1/3}-\dfrac{1}{8}n^{-2/3}+\dfrac{1}{16}n^{-1}-\cdots$

*2 $\quad (1+n)^{1/n} \sim 1+\dfrac{\log n}{n}+\dfrac{(\log n)^2}{2n^2}+\dfrac{1}{n^2}+\cdots$

Table 2. Test series

subset	partial sum of series	sum	θ
$\mathcal{L}_0$	$\sum_{j=1}^{n} \frac{2j-1}{j(j+1)(j+2)}$	0.75	-1
$\mathcal{L}_1$	$\sum_{j=1}^{n} \frac{1}{j^2}$	1.644934066848226	-1
$\mathcal{L}_2$	$\sum_{j=1}^{n} \frac{1}{j^3}$	1.202056903159594	-2
$\mathcal{L}_3$	$\sum_{j=1}^{n} \frac{1}{j^{1.5}}$	2.612375348685488	-0.5
$\mathcal{L}_4$	$\sum_{j=1}^{n} \left(\frac{1}{j^{1.5}} + \frac{1}{j^2}\right)$	4.257309415533714	(*3)
$\mathcal{L}_5$	$\sum_{j=2}^{n} \frac{\log j}{j^2}$	0.9375482543158438	(*4)

*3 $\sum_{j=1}^{n} \left(\frac{1}{j^{1.5}} + \frac{1}{j^2}\right) \sim \zeta(1.5) + \zeta(2) - 2n^{-1/2} - n^{-1} + \cdots$

*4 $\sum_{j=2}^{n} \frac{\log j}{j^2} \sim -\zeta'(2) + \frac{\log n}{n} - \frac{1}{n} + \frac{\log n}{2n^2} - \cdots,$

where $\zeta(s)$ is the Riemann zeta function and $\zeta'(s)$ is its derivative.

5. Results and conclusions.

For each acceleration method we show the maximum significant digits from 15 terms of each test sequence (partial sums of each test series) in table 3 (resp. table 4). We shall compare the diagonals in the accelerated arrays. For example, $\{\bar{\rho}_n^{(0)}\}$, $\{s_{n-j}^{j}\}$ with $j=\lfloor n/2 \rfloor$, $\{E_{n-1}^{(1)}\}$ and so on. Numerical computations have been carried out on the NEC personal computer PC-9801VM in double precision with approximately 16 digits.

Table 3. The maximum significant digits. (Group I)

		ρ-algorithm NOT	SD	aut gen ρ NOT	SD	aut mod Ait NOT	SD	gen ε-alg NOT	SD
$\mathcal{L}_0$	Q	5	15.13	15	13.36	11	13.95	13	9.87
	R	4	16.56	14	13.82	15	11.60	12	6.13
$\mathcal{L}_1$	Q	15	11.58	15	10.34	14	10.19	12	7.90
	R	14	12.49	14	12.53	15	11.58	13	8.42
$\mathcal{L}_2$	Q	13	11.81	13	11.29	13	8.10	15	7.64
	R	13	11.80	13	12.60	12	13.06	15	10.14
$\mathcal{L}_3$	Q	15	1.91	15	10.74	15	11.13	15	7.85
	R	15	1.32	14	11.03	11	10.86	12	6.96
$\mathcal{L}_4$	Q	15	1.36	15	2.79	14	2.90	6	2.50
	R	15	1.32	15	2.96	15	4.36	10	2.57
$\mathcal{L}_5$	Q	15	2.71	15	3.52	14	3.02	14	2.70
	R	15	2.73	14	3.29	12	3.35	11	3.37

		Levin u NOT	SD	Levin v NOT	SD	Lubkin W NOT	SD	θ-algorithm NOT	SD
$\mathcal{L}_0$	Q	13	10.75	13	10.26	12	10.78	15	11.16
	R	12	10.60	14	11.78	15	10.40	15	10.21
$\mathcal{L}_1$	Q	13	8.43	13	7.87	12	8.64	15	8.53
	R	13	11.09	14	11.25	15	9.13	14	11.40
$\mathcal{L}_2$	Q	14	10.86	15	10.11	13	5.50	11	6.18
	R	13	12.35	14	12.09	15	10.65	15	12.53
$\mathcal{L}_3$	Q	14	8.82	15	7.87	15	8.76	15	8.00
	R	12	10.50	13	8.97	12	9.29	15	8.59
$\mathcal{L}_4$	Q	15	2.48	15	2.72	6	2.56	14	3.34
	R	15	2.33	15	2.73	10	2.65	7	2.88
$\mathcal{L}_5$	Q	15	2.82	15	3.12	14	2.84	15	3.84
	R	15	2.76	15	3.09	10	2.13	13	2.29

NOT … number of terms

SD … significant digits

Q … sequence in Table 1

R … series in Table 2

Table 4. The maximum significant digits. (Group II)

		gen ρ-alg NOT	SD	mod Aitken NOT	SD	E-algorithm NOT	SD	Richardson NOT	SD
$\mathcal{L}_0$	Q	14	13.68	11	12.55	15	11.60	512	15.78
	R	4	16.56	15	11.95	15	9.67	2048	15.31
$\mathcal{L}_1$	Q	15	11.58	14	11.29	12	9.86	4096	15.65
	R	14	12.49	15	11.61	13	10.85	512	15.65
$\mathcal{L}_2$	Q	13	11.92	9	5.63	13	12.28	512	16.86
	R	13	12.72	14	12.80	13	12.22	256	15.41
$\mathcal{L}_3$	Q	11	13.01	14	12.87	13	10.84	1024	17.28
	R	12	11.51	12	11.57	13	10.12	2048	15.21
$\mathcal{L}_4$	Q	15	2.87	15	3.23	12	7.29	4096	11.47
	R	15	2.88	13	3.16	14	6.31	4096	12.76
$\mathcal{L}_5$	Q					15	7.26		
	R					15	10.34		

By table 3 and table 4, we conclude that superior methods for $\mathcal{L}_i$ $(i=0\sim5)$ are as follows:

(1) $\mathcal{L}_0$ … The ρ-algorithm of Wynn (exact).

(2) $\mathcal{L}_1, \mathcal{L}_2$ … The ρ-algorithm of Wynn; the generalized ρ-algorithm (more than 11 significant digits from 15 terms).

(3) $\mathcal{L}_3$ (θ is known) … The generalized ρ-algorithm; the modified Aitken δ^2-formula (more than 11 significant digits from 15 terms).

(4) $\mathcal{L}_3$ (θ is unknown) … The automatic generalized ρ-algorithm ; the automatic modified Aitken δ^2-formula (more than 10 significant digits from 15 terms).

(5) $\mathcal{L}_4$ (α_j are known)… The E-algorithm (more than 6 significant digits from 15 terms); the Richardson extrapolation (more than 11 significant digits from 4096 terms).

(6) $\mathcal{L}_4$ (α_j are unknown)… The automatic generalized ρ-algorithm; the automatic modified Aitken δ^2-formula; the generalized ε-algorithm; Levin v transform; Lubkin W transform; the θ-algorithm (more than 2.5 significant digits from 15 terms).

(7) $\mathscr{L}_5$ (asymptotic expansion is known)… The E-algorithm (more than 7 significant digits from 15 terms).

(8) $\mathscr{L}_5$ (asymptotic expansion is unknown)… The automatic generalized ρ-algorithm; the automatic modified Aitken δ^2-formula; Levin v transform (more than 3 significant digits from 15 terms).

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